

# Energy Dissipation in the Smagorinsky Model of Turbulence

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## Abstract

The Smagorinsky model, unmodified, is often reported to severely overdissipate flows. Previous estimates of the energy dissipation rate of the Smagorinsky model for *shear flows* are that

$$\langle \varepsilon_S \rangle \simeq [1 + C_S^2 \left( \frac{\delta}{L} \right)^2 (1 + Re^2)] \frac{U^3}{L}$$

reflecting a blow up of model energy dissipation as  $Re \rightarrow \infty$ . This blow up is consistent with the numerical evidence and leads to the question: *Is the over dissipation due to the influence of the turbulent viscosity in boundary layers alone or is its action on small scales generated by the nonlinearity through the cascade also a contributor?* This report develops model dissipation estimates for body force driven flow under periodic boundary conditions (and thus only with nonlinearity generated small scales). It is proven that the model's time averaged energy dissipation rate,  $\langle \varepsilon_S \rangle$ , satisfies

$$\langle \varepsilon_S \rangle \leq 3 \frac{U^3}{L} + \frac{3}{8} Re^{-1} \frac{U^3}{L} + C_S \left( \frac{\delta}{L} \right)^2 \frac{U^3}{L},$$

where  $U, L$  are global velocity and length scales and  $C_S \simeq 0.1$ ,  $\delta \ll 1$  are the standard model parameters. Since this estimate is consistent with that observed for the NSE, it establishes that, *without boundary layers, the Smagorinsky model does not over dissipate.*

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# 1 Introduction

*This is an expanded version of a report with a similar title.*

The Smagorinsky model<sup>1</sup>, from [S63] and used for of turbulent flow, e.g., [BIL06], [S01], [J04], [M97], [P94], [G97], is

$$u_t + u \cdot \nabla u - \nu \Delta u - \nabla \cdot \left( (C_S \delta)^2 |\nabla u| \nabla u \right) + \nabla p = f(x) \quad (1)$$

and  $\nabla \cdot u = 0$ .

It is mathematically equivalent to the Ladyzhenskaya model<sup>2</sup> [S63], [L67], [DG91], [P92] and the von Neumann Richtmyer artificial viscosity for shocks [vNR50]. In (1),  $\nu$  is the kinematic viscosity,  $\delta \ll 1$  is a model length scale, the Reynolds number is

$$Re = \frac{LU}{\nu}$$

where  $U, L$  denote global velocity and length scales given by (3) below and  $C_S$  is a model parameter. See [S84] for ranges of values around the value  $C_S \simeq 0.1$  determined by Lilly [L67]. Experience with the model, e.g., [S01], [BIL06], strongly suggests it over dissipates, often severely (see Section 3 for some fixes). Estimates of model energy dissipation rates for shear flows in [L02] are consistent with this computational experience. Perhaps surprisingly, herein we show that the time averaged energy dissipation rate for (1) balances the energy input rate<sup>3</sup>,  $U^3/L$ . Thus *the Smagorinsky model does not over dissipate energy for body force driven turbulence in a periodic box*. In other words, the observed over dissipation of the model is not due to the model's action on small scales generated by the nonlinear term in the turbulent cascade but rather it is *due to the action of the model viscosity in boundary layers*.

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<sup>1</sup>In its most precise realization, the term  $\nabla \cdot \left( (C_S \delta)^2 |\nabla u| \nabla u \right)$  is replaced by  $\nabla \cdot \left( 2 (C_S \delta)^2 |\nabla^s u| \nabla^s u \right)$  where  $\nabla^s u := (\nabla u + \nabla u^T)/2$  is the deformation tensor. The analysis herein holds by the same argument.

<sup>2</sup>In the Smagorinsky model one has the viscous terms  $-\nu \Delta u - \nabla \cdot \left( (C_S \delta)^2 |\nabla u| \nabla u \right)$  while in the Ladyzhenskaya model the corresponding terms are  $-\nabla \cdot \left( \sqrt{\nu^2 + (C_S \delta)^4} |\nabla u|^2 \nabla u \right)$ . For both, the most precise realization replaces the velocity gradient with the deformation tensor.

<sup>3</sup>The energy input rate at the large scales is  $U^3/L$ . Briefly, the kinetic energy of the large scales scales with dimensions  $U^2$ . The "rate" has dimensions  $1/time$ . A large scale quantity with this dimensions is formed by  $U/L$  which is the turn over time for the large eddies, i.e., the time it takes a large eddy with velocity  $U$  to travel a distance  $L$ . Thus the "rate of energy input" has dimensions  $U^3/L$ .

*How does the fluid velocity produced from a numerical simulation on a fixed grid communicate with molecular viscosity?* - J. Smagorinsky 1960

The motivation for the model can be described in a very general way as follows. For high Reynolds numbers, dissipation occurs non-negligibly only at very small scales, far smaller than typical meshes. The balance between energy input at the largest scales and energy dissipation at the smallest is a critical selection mechanism for determining statistics of turbulent flows. Joseph Smagorinsky was concerned with geophysical flow simulations and soon asked the above question. Somehow, once a mesh is selected, to get accurate simulations extra dissipative terms must be introduced to model the effect of the unresolved fluctuations (smaller than the mesh width) upon the resolved velocity (representable on the mesh). Thus the extra term was in (1) was introduced where the length scale  $\delta$  was intended to reflect the underlying physical mesh length.

Let  $\Omega = (0, L_\Omega)^3$  denote the periodic box in  $3d$  and impose periodic (with zero mean) conditions

$$\begin{aligned} u(x + L_\Omega e_j, t) &= u(x, t) \quad j = 1, 2, 3 \text{ and} \\ \int_\Omega \phi dx &= 0 \quad \text{for } \phi = u, u_0, f, p. \end{aligned} \tag{2}$$

The data  $u_0(x), f(x)$  are smooth,  $L_\Omega$ -periodic, have zero mean and satisfy

$$\nabla \cdot u_0 = 0, \text{ and } \nabla \cdot f = 0.$$

The model energy dissipation rate from (5) below is

$$\varepsilon_S(u) := \int_\Omega \frac{\nu}{|\Omega|} |\nabla u(x, t)|^2 + \frac{(C_S \delta)^2}{|\Omega|} |\nabla u(x, t)|^3 dx.$$

The long time average of a function  $\phi(t)$  is defined, following [DF02], [DG95] by

$$\langle \phi \rangle := \lim_{T \rightarrow \infty} \sup \frac{1}{T} \int_0^T \phi(t) dt.$$

We show herein that  $\langle \varepsilon_S \rangle$  balances the energy input rate,  $U^3/L$ . This estimate is consistent as  $Re \rightarrow \infty, \delta \rightarrow 0$  with both phenomenology, e.g., [F95], [Po00], [M97], and the rate proven for the Navier-Stokes equations in [CD92], [CKG01], [W97], [DF02] and [CDP06]. The weak  $Re$  dependence in the second term (that vanishes as  $Re \rightarrow \infty$ ) is consistent with the recent results in [MBYL15] derived through structure function theories of turbulence.

**Theorem 1** *Suppose the data  $f(x)$  and  $u_0(x)$  are smooth, divergence free, periodic with zero mean functions. Then*

$$\langle \varepsilon_S(u) \rangle \leq 3 \frac{U^3}{L} + \frac{3}{8} Re^{-1} \frac{U^3}{L} + C_S^2 \left( \frac{\delta}{L} \right)^2 \frac{U^3}{L}$$

## 1.1 Improving the constant multiplier

The multiplier "3" of  $\frac{U^3}{L}$  can be reduced to  $1/(1-\alpha)$  for any  $\alpha > 0$ , arbitrarily close to "1", at the cost of a multipliers  $\alpha^{-1}$  and  $\alpha^{-2}$  of the other two terms on the RHS by inserting parameters at various points in the argument. The following is the precise result.

**Theorem 2**  $\langle \varepsilon_S \rangle$  satisfies: for any  $0 < \alpha < 1$ ,

$$\langle \varepsilon_S(u) \rangle \leq \frac{1}{1-\alpha} \frac{U^3}{L} + \frac{1}{4\alpha(1-\alpha)} \text{Re}^{-1} \frac{U^3}{L} + \frac{4}{27(1-\alpha)\alpha^2} C_S^2 \left( \frac{\delta}{L} \right)^2 \frac{U^3}{L}.$$

## 1.2 Related work

The energy dissipation rate is a fundamental statistic in experimental and theoretical studies of turbulence, e.g., Sreenivasan [S84], Pope [Po00], Frisch [F95], Lesieur [L97]. In 1968, Saffman [S68], addressing the estimate of energy dissipation rates,  $\langle \varepsilon \rangle \simeq U^3/L$ , wrote that

*"This result is fundamental to an understanding of turbulence and yet still lacks theoretical support."* - P.G. Saffman 1968

In 1992 Constantin and Doering [CD92] made a fundamental breakthrough, establishing a direct link between the phenomenology of energy dissipation and that predicted for general weak solutions of shear flows directly from the NSE. This work builds on Busse [B78], Howard [H72] (and others) and has developed in many important directions. It has been extended to shear flows in Childress, Kerswell and Gilbert [CKG01], Kerswell [K98] and Wang [W97]. For flows driven by body forces extensions include Doering and Foias [DF02], Cheskidov, Doering and Petrov [CDP06] (fractal body forces), and [L07] (helicity dissipation). The energy dissipation rates of discretized (so the smallest scale is limited by the mesh width and time step) flow equations was studied in [JLM07], [JLK14]. Energy dissipation in models and regularizations studied in [L02], [L07], [LRS10], [LST10]. Most recently, the time averaged energy dissipation in statistical fluctuations has led to new models and a proof of the Boussinesq conjecture in [JLK14],[JL15].

## 2 The proof

Let  $\|\cdot\|, (\cdot, \cdot)$  denote the usual  $L^2(\Omega)$  norm and inner product. Other norms are explicitly indicated by a subscript. With  $|\Omega|$  the volume of the flow domain, the scale of the body force, large scale velocity and length,  $F, U, L$ , are defined

by

$$\begin{aligned}
F &= \left( \frac{1}{|\Omega|} \|f\|^2 \right)^{\frac{1}{2}}, \\
U &= \left\langle \frac{1}{|\Omega|} \|u\|^2 \right\rangle^{\frac{1}{2}}, \\
L &= \min \left\{ |\Omega|^{\frac{1}{3}}, \frac{F}{\|\nabla f\|_{L^\infty}}, \frac{F}{(\frac{1}{|\Omega|} \|\nabla f\|^2)^{\frac{1}{2}}}, \frac{F}{(\frac{1}{|\Omega|} \|\nabla f\|_3^3)^{\frac{1}{3}}} \right\}.
\end{aligned} \tag{3}$$

It is easy to check that  $L$  has units of length and satisfies the inequalities:

$$\left. \begin{aligned} \|\nabla f\|_{L^\infty} &\leq \frac{F}{L}, \\ \frac{1}{|\Omega|} \int_{\Omega} |\nabla f(x)|^2 dx &\leq \frac{F^2}{L^2}, \\ \frac{1}{|\Omega|} \int_{\Omega} |\nabla f(x)|^3 dx &\leq \frac{F^3}{L^3}. \end{aligned} \right\} \tag{4}$$

The proof is a synthesis of the model's energy balance (5), the breakthrough arguments of Doering and Foias [DF02] from the NSE case with careful treatment of the Smagorinsky term.

Solutions to the Smagorinsky / Ladyzhenskaya model are known, e.g., [C98], [DG91], [L69], [L67], [P92] [G89], to be unique strong solutions and satisfy the energy equality

$$\frac{1}{2|\Omega|} \|u(T)\|^2 + \int_0^T \varepsilon_S(u) dt = \frac{1}{2|\Omega|} \|u_0\|^2 + \int_0^T \frac{1}{|\Omega|} (f, u(t)) dt. \tag{5}$$

Here  $\varepsilon_S(u) = \varepsilon_0(u) + \varepsilon_\delta(u)$ , where

$$\begin{aligned}
\varepsilon_0(u) &:= \frac{\nu}{|\Omega|} \|\nabla u(t)\|^2 \text{ and} \\
\varepsilon_\delta(u) &:= \frac{(C_S \delta)^2}{|\Omega|} \|\nabla u(t)\|_{L^3}^3.
\end{aligned}$$

From (5) and standard arguments it follows that

$$\begin{aligned}
\sup_{t \in (0, \infty)} \|u(t)\|^2 &\leq C(\text{data}) < \infty \text{ and} \\
\frac{1}{T} \int_0^T \varepsilon_S(u) dt &\leq C(\text{data}) < \infty.
\end{aligned} \tag{6}$$

Averaging (5) over  $[0, T]$ , applying the Cauchy-Schwarz inequality in time and (6) yields

$$\begin{aligned}
\frac{1}{T} \int_0^T \varepsilon_S(u) dt &= \mathcal{O}\left(\frac{1}{T}\right) + \frac{1}{T} \int_0^T \frac{1}{|\Omega|} (f, u(t)) dt \\
&\leq \mathcal{O}\left(\frac{1}{T}\right) + F \left( \frac{1}{T} \int_0^T \frac{1}{|\Omega|} \|u\|^2 dt \right)^{\frac{1}{2}}.
\end{aligned} \tag{7}$$

To bound the RHS, take the inner product of (1) with  $f(x)$ , integrate by parts and average over  $[0, T]$ . This gives

$$\begin{aligned} F^2 &= \frac{(u(T) - u_0, f)}{T|\Omega|} - \frac{1}{T} \int_0^T \frac{1}{|\Omega|} (uu, \nabla f) dt + \\ &+ \frac{1}{T} \int_0^T \frac{\nu}{|\Omega|} (\nabla u, \nabla f) dt + \frac{1}{T} \int_0^T \frac{1}{|\Omega|} \left( (C_S \delta)^2 |\nabla u| \nabla u, \nabla f \right) dt. \end{aligned} \quad (8)$$

Of the four terms on the last RHS, by (6) the first term is  $\mathcal{O}(1/T)$ . The second and third terms are bounded using the Cauchy-Schwarz-Young inequality and (4) by

$$\begin{aligned} \left| \frac{1}{T|\Omega|} \int_0^T (uu, \nabla f) dt \right| &\leq \|\nabla f\|_{L^\infty} \frac{1}{T|\Omega|} \int_0^T \|u\|^2 dt \\ &\leq \frac{F}{L} \left( \frac{1}{T} \int_0^T \frac{1}{|\Omega|} \|u\|^2 dt \right), \\ \left| \frac{1}{T} \int_0^T \frac{\nu}{|\Omega|} (\nabla u, \nabla f) dt \right| &\leq \left( \frac{1}{T} \int_0^T \frac{\nu^2}{|\Omega|} \|\nabla u\|^2 dt \right)^{\frac{1}{2}} \left( \frac{1}{T} \int_0^T \frac{1}{|\Omega|} \|\nabla f\|^2 dt \right)^{\frac{1}{2}} \\ &\leq \left( \frac{1}{T} \int_0^T \varepsilon_0(u) dt \right)^{\frac{1}{2}} \sqrt{\nu} \frac{F}{L} \\ &\leq \frac{2}{3} U^{-1} F \frac{1}{T} \int_0^T \varepsilon_0(u) dt + \frac{3}{8} U F \frac{\nu}{L^2}. \end{aligned}$$

The fourth, Smagorinsky, term, is estimated using Hölder's inequality as follows

$$\begin{aligned} \left| \frac{1}{T|\Omega|} \int_0^T \left( (C_S \delta)^2 |\nabla u| \nabla u, \nabla f \right) dt \right| &\leq \frac{(C_S \delta)^2}{|\Omega|} \frac{1}{T} \int_0^T \|\nabla u\|_{L^3}^2 dt \|\nabla f\|_{L^3} \\ &\leq \frac{F}{L} (C_S \delta)^{2/3} \frac{1}{T} \int_0^T \varepsilon_\delta(u)^{\frac{2}{3}} dt. \end{aligned}$$

Insert multipliers of  $U^{2/3}$  and  $U^{-2/3}$  in the two terms. Using<sup>4</sup>

$$ab \leq (2/3)a^{3/2} + (1/3)b^3$$

(conjugate exponents 3/2 and 3) gives

$$\frac{1}{T} \int_0^T \left( \frac{U^{2/3}}{L} (C_S \delta)^{2/3} \right) \left( U^{-2/3} \varepsilon_\delta(u)^{\frac{2}{3}} \right) dt \leq \frac{2}{3} \frac{1}{U} \frac{1}{T} \int_0^T \varepsilon_\delta(u) dt + \frac{U^2}{3} \frac{(C_S \delta)^2}{L^3}.$$

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<sup>4</sup>More generally, for conjugate  $(1/p + 1/q = 1)$  exponents and  $a > 0, b > 0$ :  $ab \leq (1/p)a^p + (1/q)b^q$ .

Using these three estimates in (8) yields

$$F \leq \mathcal{O}\left(\frac{1}{T}\right) + \frac{1}{L} \frac{1}{T} \int_0^T \frac{1}{|\Omega|} \|u\|^2 dt + \\ + \frac{3}{8} \frac{U\nu}{L^2} + \frac{2}{3} \frac{1}{U} \frac{1}{T} \int_0^T \varepsilon_S(u) dt + \frac{U^2}{3} \frac{(C_S \delta)^2}{L^3}.$$

Using this estimate for  $F$  in (7) gives

$$\frac{1}{T} \int_0^T \varepsilon_S(u) dt \leq \mathcal{O}\left(\frac{1}{T}\right) + F \left( \frac{1}{T} \int_0^T \frac{1}{|\Omega|} \|u\|^2 dt \right)^{\frac{1}{2}} \\ \leq \mathcal{O}\left(\frac{1}{T}\right) + \left( \frac{1}{T|\Omega|} \int_0^T \|u\|^2 dt \right)^{\frac{1}{2}} \times \\ \left( \frac{1}{LT|\Omega|} \int_0^T \|u\|^2 dt + \frac{3}{8} \frac{U\nu}{L^2} + \frac{2}{3} \frac{1}{UT} \int_0^T \varepsilon_S(u) dt + \frac{U^2}{3L^3} (C_S \delta)^2 \right).$$

Taking the limit superior, which exists by (6), as  $T \rightarrow \infty$  we obtain

$$\langle \varepsilon_S(u) \rangle \leq \frac{U^3}{L} + \frac{3}{8} \frac{U^3}{L} \frac{\nu}{LU} + \frac{2}{3} \langle \varepsilon_S(u) \rangle + \frac{U^3}{3L} \frac{(C_S \delta)^2}{L^2}.$$

Thus, as claimed,

$$\langle \varepsilon_S(u) \rangle \leq 3 \frac{U^3}{L} + \frac{9}{8} Re^{-1} \frac{U^3}{L} + \frac{U^3}{L} C_S^2 \left( \frac{\delta}{L} \right)^2.$$

### 3 Conclusions for the Smagorinsky Model

Comparing the estimate

$$\langle \varepsilon_S \rangle \simeq \frac{U^3}{L}$$

herein for periodic boundary conditions with

$$\langle \varepsilon_S \rangle \simeq [1 + C_S^2 \left( \frac{\delta}{L} \right)^2 (1 + Re^2)] \frac{U^3}{L}$$

in [L07] for shear flows with boundary layers strongly suggests the often reported model over dissipation is

*due to the action of the model viscosity in boundary layers*

rather than in interior small scales generated by the turbulent cascade. Practice addresses this over dissipation with damping functions (e.g., van Driest damping [vD12],[S01]), modelled boundary conditions called near wall models,

e.g., [PB02], [JLS04], [JL06], restriction of the model induced dissipation to the smallest resolved scales [HOM01] and Germano's dynamic (self-adaptive) selection of  $C_S = C_S(x, t)$ , [GPMC91], that also reduces  $C_S$  near walls. Thus, analysis of Smagorinsky model dissipation for shear flows *including these modifications* is therefore an important open problem.

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